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Analytical calculation of flow profiles and forces between teatcup and teat

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Abstract

It can be observed that the teat cups migrate up the teat during milking process. To explain this observation, it is supposed that there are short time intervals during the milking process where the positive locking between the teat and teatcup does not exist. In this case, a flow is formed between the teat and teat rubber which generates a force.
The force is considered on the basis of the flow between teat and inner wall of the teat cup depending on various parameters. It is necessary to calculate and study the flow profile in this area. The flow profile is calculated by the Navier-Stokes equations in cylindrical coordinates. With the assumptions that the flow is stationary, the velocity components in r- and phi- direction are zero and the flow is cylindrically. The solution of this equation gives us the velocity in z-direction depending on the pressure, the radius and two more variables. These variables can be calculated by using the no-slip condition on the boundary surface, which says that the flow rate of a viscous fluid at a solid boundary surface is equal to zero or is zero relative to the velocity of the boundary surface. In the size range of common teatcup rubber the result is nearly a square flow profile.
Finally the force can be considered by a flow around body based on the viscosity of the fluid flowing around it. The calculation has to be done for the force on the inner wall of the concentric cylinders. The force on the flow around surface of a cylinder is given by an equation depending of the cylinder radius, length, the viscosity of the flowing material and the velocity. The velocity profile otherwise depends on the pressure difference. But with some analytical transitions it can be shown that the force only depends on the flow rate which can be calculated from the average of the velocity profile and the cross-sectional flow area. In result of the work, the acting force at the teat can be calculated for a given pressure difference.

Keywords: flow profile, pressure, force, Navier-Stokes-

1 Introduction

The force is considered on the basis of the flow between teat and inner wall of the teat cup depending on various parameters. It is necessary to calculate and study the flow profile in that area. The teat and the teat rubber are assumed to be concentric cylinders.

2 Methods

2.1 Calculation of flow profiles

2.1.1 Flow in the space between two concentric cylinders

The flow profile is calculated by the Navier-Stokes equations. A transformation of these equations in cylindrical coordinates r, φ and z leads to

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\phi^2}{r}\right)=\frac{-\partial p}{\partial r}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2}+\frac{\partial_r^{2v}}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}\right]$$

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\phi^2}{r}\right)=\frac{-1}{r}\frac{\partial p}{\partial \phi}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2}+\frac{\partial^2 v_\phi}{\partial z^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \phi}-\frac{v_\phi}{r^2}\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi}+v_z\frac{\partial v_z}{\partial z}\right)=\frac{-\partial p}{\partial z}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2}+\frac{\partial^2 v_z}{\partial z^2}\right] .$$

In the next step, the flow is assumed to be a Hagen-Poiseuille flow. As a result, the following assumption can be made.

1. The flow is stationary (all time derivatives are zero).
2. The velocity components in r- and φ - direction are zero.
3. The flow is cylindrically symmetric and fully developed.

With these assumptions, the equation simplifies to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)=\frac{1}{\eta}\frac{\partial p}{\partial y} . \tag{1}$$

The soltuion of this differential equation is

$$v_z=\frac{1}{4\eta}\frac{\partial p}{\partial z}r^2+a\cdot\ln\left(r/R_o\right)+b . \tag{2}$$

By taking into account of the no-slip condition on the boundary surfaces exist two conditions and with that aid the constants a and b can be calculated.
The no-slip condition states that the flow rate of a viscous fluid at a solid boundary surfaces equal to zero or is zero relative to the velocity of the boundary surfaces. For the constants a and b results

$$b=\frac{-1}{4\eta}\frac{\partial p}{\partial z}R_o^2 \tag{3}$$

and

$$a=\frac{1}{4\eta}\frac{\partial p}{\partial z}\frac{1}{\ln\left(R_i/R_o\right)}\left(R_o^2-R_i^2\right) . \tag{4}$$

Assuming a linear pressure drop along the flow direction it is valid that the partial derivative $\frac{\partial p}{\partial z}$ is equal to $\frac{\Delta p}{\Delta z}=\frac{p_1-p_2}{z_1-z_2}$. The function of the flow profile is given by

$$v_z\left(\Delta p,\Delta z,\eta,R_i,R_o,r\right)=\frac{1}{4\eta}\frac{\Delta p}{\Delta z}r^2+\frac{1}{4\eta}\frac{\Delta p}{\Delta z}\frac{R_o^2-R_i^2}{\ln\left(R_i/R_o\right)}\cdot\ln\left(r/R_a\right)-\frac{1}{4\eta}\frac{\Delta p}{\Delta z}R_A^2 \tag{5}$$

or rather

$$v_z\left(\Delta p, \Delta z, \eta, R_i, R_o, r\right)=\frac{1}{4\eta}\frac{\Delta p}{\Delta z}\left(r^2+\frac{R_o^2-R_i^2}{\ln\left(R_i/R_o\right)}\cdot\ln\left(r/R_a\right)-R_A^2\right) \quad . \tag{6}$$

For closer consideration of the equation, the limit is determined by

$$\lim_{R_i \to R_o}\frac{R_o^2-R_i^2}{\ln\left(R_i/R_o\right)} \tag{7}$$

Since the limits $\lim_{R_i \to R_o} R_o^2-R_i^2$ and $\lim_{R_i \to R_o} \ln\left(R_i/R_o\right)$ both are equal to zero, the rule of L'Hospital can be used to calculate the limit. Therefore it is

$$\lim_{R_i \to R_o}\frac{R_o^2-R_i^2}{\ln\left(R_i/R_o\right)}=\lim_{R_i \to R_o} -R_i^2=-2\mathrm{R}_o^2 \tag{8}$$

With the limit value $\lim_{r \to R_A} \ln\left(r/R_o\right)=0$ is shown that the influence of the second term in equation 6 is valid only by a small inner radius R_i and a large outer radius R_o. In the size range of common teatcup rubber, the second term in that equation has a less influence. The result is a nearly square flow profile. In figure 1 is shown the solution curve of equation 6 at the point $\Delta z=70\mathrm{mm}$ for a pressure-difference $\Delta p=-7kPa$, an outer radius R_o=15,5mm and an inner radius R_i=15mm plotted as a function of the radius r. The flow profile is similar to the profile between two parallel plates.

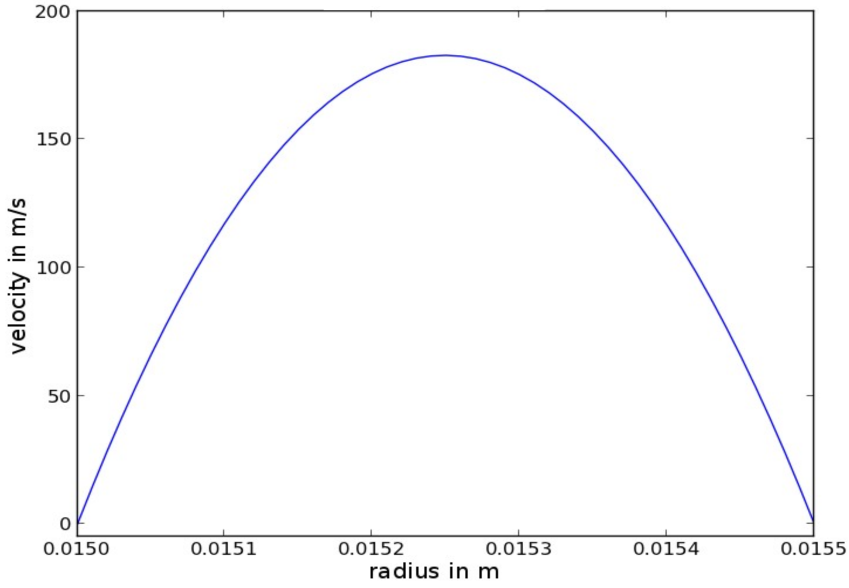

Figure 1: flow profiles between two concentric cylinders

2.2 Force due to viscous flow

In this part the force is to be considered by a flow around body on the basis of viscosity of the fluid flowing around it. The force applies to a flow-around surface is

$$F=\eta A\frac{\partial v}{\partial x} \quad . \tag{9}$$

2.2.1 Forces on an overflowed cylinder

In this example the force is examined for the inner wall of the concentric cylinders as described in the section before. The force on the flow around surface of a cylinder is given by

$$F=\eta 2\pi R_{cylinder} l \frac{\partial v}{\partial x} \quad . \tag{10}$$

Here is $R_{cylinder}$ the radius and l the length of the cylinder. For calculating the force the derivative of the velocity at the point R_i must be determined. The derivation of equation 6 is

$$\frac{\partial v_z}{\partial r}=\frac{\Delta p}{4\eta\Delta z}(2r+\frac{R_o^2-R_i^2}{\ln(R_i/R_o)}\frac{1}{r}) \quad . \tag{11}$$

Applies the force to the inner cylinder

$$F=\eta 2\pi R_i l \frac{\Delta p}{4\eta l}\left(2R_i+\frac{R_o^2-R_i^2}{\ln\left(R_i/R_o\right)}\frac{1}{R_i}\right) \tag{12}$$

or rather

$$F=\pi\frac{\Delta p}{2}\left(2R_i^2+\frac{R_o^2-R_i^2}{\ln\left(R_i/R_o\right)}\right) \quad . \tag{13}$$

By equation 13 it can be seen that in the stationary case, the force is not depending by length of the cylinder and also not by the viscosity of the medium flowing around it.

3 Results and Discussions

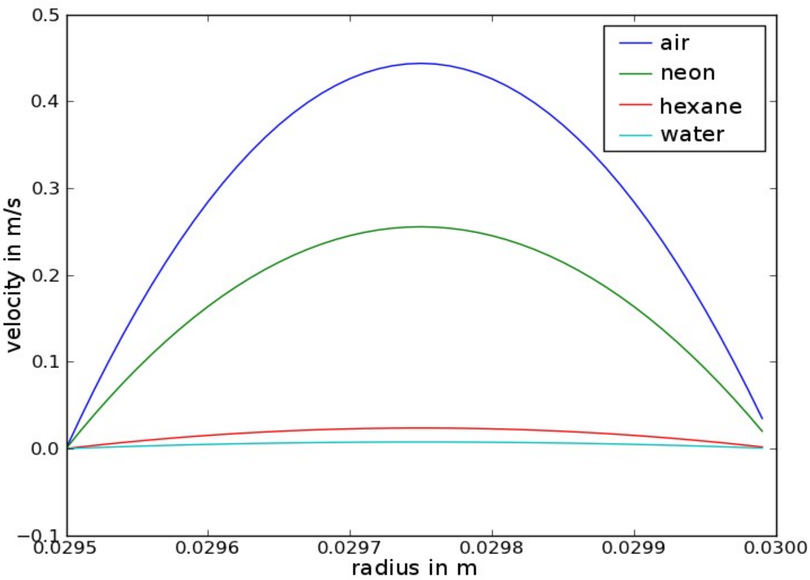

Figure 2: flow profiles of different elements

Element	Viscosity
air	17.1 $\mu Pa\ s$
neon	29.7 $\mu Pa\ s$
hexane	0.320 $m Pa\ s$
water	1 $m Pa\ s$

Table 1: viscosity of plotted elements

To explain the seeming independence of the force of the viscosity the flow profile of various viscous substances are illustrated in figure 2. The pressure difference is always $\Delta p = 17\,kPa$. The viscosity of the individual substances is listed in chart 1. At otherwise constant parameters, more viscous substances forms flatter flow profiles. In the original equation for force due to viscous flow $F = \eta \cdot A \frac{v}{x}$, the force depends for example from the viscosity and a given velocity. For getting the same flow profile by constant flow length Δz with a substance A and viscosity η_A and a substance B with viscosity $\eta_B = \frac{\eta_A}{2}$, the pressure difference Δp_A must be twice the pressure difference Δp_B .
According to equation 13 the force depends from the increasing of the flow profile at the point R_i and the pressure difference Δp .
This means that for substances with higher viscosity η the pressure difference Δp must be greater to get same flow profile like subtances with lower viscosity η , so that the force is greater at higher viscose materials.
The figure 3 shows the force as a function of the pressure difference. The inner radius R_i=29,5mm and the outer radius R_o=30mm are constant. The equation shows that there exist a linear connection in this case.

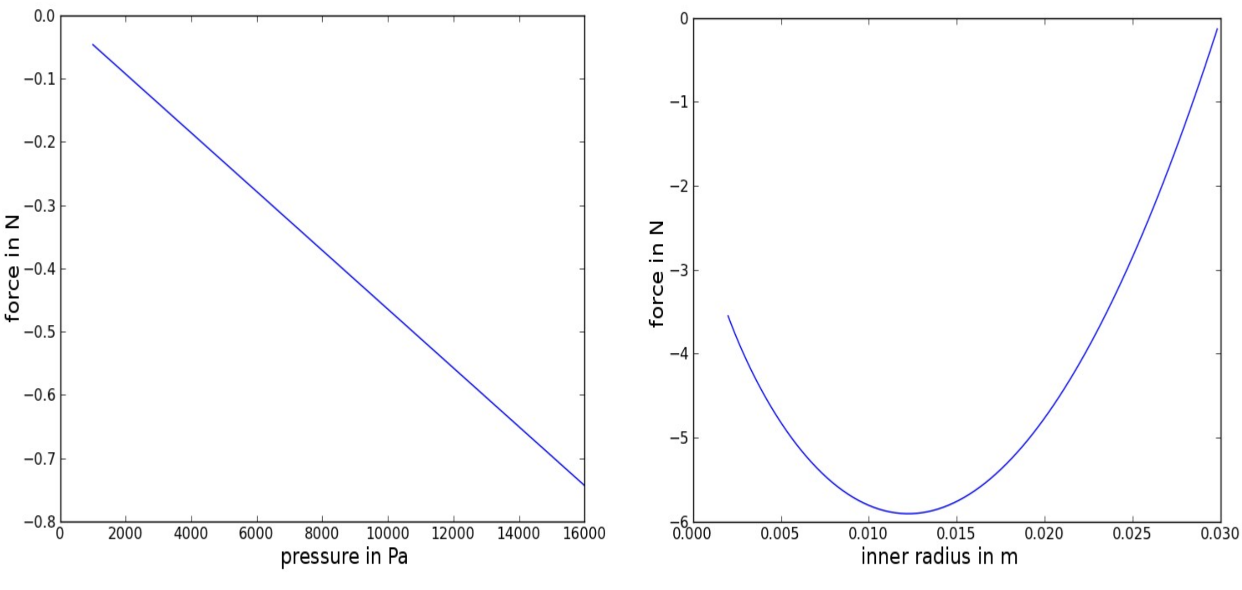

Figure 3: force as a function of the pressure difference (left) and force as a function of the inner cylinder radius (right)

In a further analysis (Figure 3 right) the pressure difference is constant at $\Delta p = 7\,kPa$ and the outside radius is R_o=30mm. So the force can examined as a function of various differences R_o-R_i. It can be seen that there is an extreme value in the range of R_i=12 mm. Analytically the extreme value results in the zeropoint of the derivative of equation 13 by the inner radius R_i.

$$\frac{dF}{dR_i}=\pi\,\Delta p\left(2R_i-\frac{-R_i}{\ln\left(R_i/R_o\right)}-\frac{R_A^2-R_i^2}{2\ln\left(R_i/R_o\right)^2 R_i}\right)=0 \qquad (14)$$

With the adopted values $\Delta p=7\,kPa$ and R_o=30mm in figure 3 the extreme value is at a force of F=-5.9 N for an inner radius R_i=12.2 mm.

Another approach to analysis the force is the observation as a function of the flow rate $Q=\frac{dV}{dt}$. The flow rate can be calculated by the flow velocity and the cross-sectional flow area.

$$Q=v\cdot A \qquad (15)$$

In the considered case by a flow between two concentric hollow cylinders, the cross-sectional area corresponds to a circular ring.

$$A=\pi\cdot\left(R_o^2-R_i^2\right) \qquad (16)$$

Since the flow velocity is not constant on this surface, the volume flow is approximated with an average velocity $\bar{v}$. For the following calculations is not the arithmetic average of the velocity important. Necessary is the square average.

$$\bar{v}=\sqrt{\frac{1}{R_o-R_i}\int_{R_i}^{R_o} v\left(r\right)^2 dr} \qquad (17)$$

According to the first part (equation 6) the results $a=\frac{1}{4\eta}\frac{\Delta p}{\Delta z}$ and $b=\frac{R_o^2-R_i^2}{\ln\left(R_i/R_o\right)}$ are used in equation 18. For the flow profile in the intermediate space between two concentric cylinders, the average flow rate can be determined by

$$\bar{v}=\sqrt{\frac{a}{\left|R_o-R_i\right|}\int_{R_i}^{R_o} R_o^4-2R_o^2r^2-2bR_o^2\ln\left(\frac{r}{R_A}\right)+r^4+2br^{2\ln}\left(\frac{r}{R_o}\right)+b^2\ln^2\left(\frac{r}{R_o}\right)dr} \quad . \qquad (18)$$

Parameter	Value
η	$17.1\,\mu Pa\ s$
R_i	$29.5\ mm$
R_o	$30\ mm$
Δp	$7\ kPa$
Δz	$70\ mm$

Table 2: used values to calculate the average flow velocity

Assuming the values listed in table 2, the calculation of the average flow speed is $\bar{v}=133.46\frac{m}{s}$.

The result is a volume flow of $Q=0.01247\frac{m^3}{s}=12.47\frac{dm^3}{s}$. If the flow rate is hold constant and the radius of the outer cylinder is reduced gradually to the radius of the inner cylinder, the flow rate between the the two cylinders will increase. By higher velocities the force increase on the inner cylinder.

$$F=\eta\,2\pi\,R_i l\frac{\bar{v}}{l}=\eta\,2\,R_i\frac{Q}{\left(R_o^2-R_i^2\right)} \qquad (19)$$

4 Conclusions

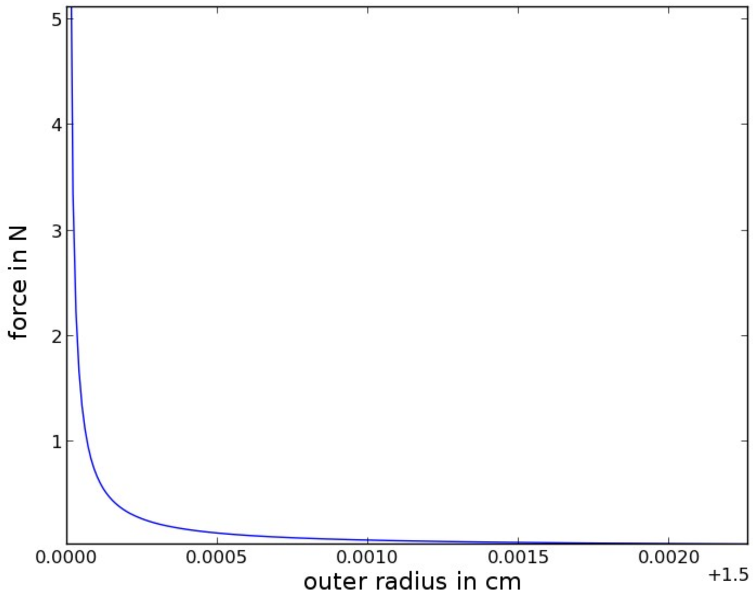

Figure 4: Force at constant volume flow as a function of the cylinder radius

Figure 4 illustrates the behavior of the function according to equation 19. If at constant volume flow R_o is running to R_i the flow rate and the force (on basis of the proportional connection) must strive towards infinity. If the total weight of a teat cup will be compensated by this force, the distance a between the cylinders could calculated by the following equation.

$$a = R_o - R_i = \sqrt{\frac{2\eta R_i Q}{F_G} + R_i^2} - R_i \qquad (20)$$

With the suppose values a value of a=40nm had been estimated.

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